Symmetric Input-Output Tables: Products or Industries?

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ABSTRACT: The recent revival of input-output analysis in trade, environmental, and productivity studies—facilitated by frequent OECD publications, comes with a controversy on the construction and use of product versus industry tables. This paper shows that the issue emerges at two levels. Product-by-product tables and industry-by-industry tables co-exist and each type can be constructed according to a product technology model or an industry model. Most countries adhere to the U.N. (1993) sanctioned theory of Kop Jansen and ten Raa (1990) and construct product technology model based product-by-product tables, but a few hard to neglect countries dissent. This paper shifts attention from theory to empirics and provides encompassing formulas that admit testing of the competing models.

KEYWORDS: input-output tables

1. Introduction

Input-output coefficients measure the inputs required per units of outputs. It makes a difference if the inputs and outputs are products or industry deliveries. Product-by-product tables are conceptually clean and their construction has nice theoretical foundations. Industry-by-industry tables make a come-back though (Yamano and Ahmad, 2006). We will show that *both* types of symmetric input-output tables (SIOTs) can be constructed by a product-technology model or an industry-technology model, although the models are slightly different for the two types.

For product-by-product tables the product-technology model is the favorite, on theoretical grounds (Kop Jansen and ten Raa, 1990 and ten Raa and Rueda Cantuche, 2003) and in terms of country adoption. The industry technology model has advantages too (the coefficients are not negative, Rueda-Cantuche, 2007) and is adopted by few but hard to neglect countries, namely Canada, Denmark, Finland, the Netherlands, and Norway. Recently we have launched an idea to settle the controversy on empirical grounds (ten Raa and Rueda Cantuche, 2007) and in this paper we follow up with a test. We will also show that the situation is the opposite for industry-by-industry tables: There the industry technology model is the natural one but may produce negatives, while product technology model coefficients are nonnegative. A test for this context will be offered as well.

Our approach to the choice of model in the construction of input-output tables—be they product-by-product or industry-by-industry—is user friendly. We will show that the choice is a matter of alternative transfer procedures for secondary outputs and their inputs—be they products or industry deliveries. The alternatives are represented by different specifications of two general transfer formulas (one for product-by-product tables and one for industry-by-industry tables). The tests flow naturally from this approach. One disclaimer is in order. We make no recommendations on the choice between product-by-product industry-by-industry tables. That seems to us a matter of neither axioms nor tests, but of scope of applications.

2. Transfers

The data consist of use matrix $\mathbf{U} = (u_{ij})$, comprising products i (= 1, …, n) consumed by industries j (= 1, …, n), and a supply matrix $\mathbf{V} = (v_{ij})$ (formerly the transposed of a make matrix) comprising the products i produced by the industries j.

The construction of a product-by-product SIOT is indicated by figure 1. Starting from the intermediate supply table, the column vector of industry j comprises the primary outputs (product j) plus the secondary outputs (the blue circles). The secondary outputs are transferred out to the industries for which they are primary outputs and the products j produced by other industries (as secondary activities) are transferred in.

PLACE HERE FIGURE 1

Figure 2 depicts the construction of an industry-by-industry SIOT. Now total outputs of industries consist of both primary and secondary activities and all the secondary elements from the column vector of industry j are transferred to the j-th main diagonal element. Similarly, products j produced elsewhere are transferred to the industries that produce them as secondary products.

PLACE HERE FIGURE 2

When transferring outputs, the corresponding inputs must be transferred along. There are alternative ways to decide how much input corresponds with output. Now a flexible framework to address this issue is given by input-output coefficients with *three* subscripts. The first subscript indexes the *input*, the second the *observation* unit, and the third the *output*. Product-by-product coefficients will be de noted by a and industry-by-industry coefficients by b. *Product coefficient* a_{ijk} is defined as the amount of product i used by industry j to make a unit of product k. *Industry coefficient* b_{jik} is defined as the delivery by industry j in product market i per unit of output of industry k.

The point of departure for the construction of a SIOT is the intermediate supply u_{ij}, the amount of product i used by industry j. When we construct a *product-by-product* table, we realize that the intermediate supply is not exclusively destined for the main product of industry j and transfer out the secondary products v_{kj}, $k \neq j$, and their input requirements, $a_{ijk}v_{kj}$, from industry j to industry k. The flipside of the coin is that amounts $a_{ikj}v_{jk}$ are transferred in from industries k, $k \neq j$. After these transfers the amount of product i used to make product j is:

$$u_{ij} - \sum_{k\neq j} a_{ijk}v_{kj} + \sum_{k\neq j} a_{ikj}v_{jk} \qquad (1)$$

When we construct an *industry-by-industry* table, we realize that the intermediate supply does not originate exclusively from the main supplier of product i and transfer out the secondary supplies v_{ik}, $k \neq i$, and their input requirements, $b_{jik}v_{ik}$, from market i to market k. The flipside of the coin is that amounts $b_{jki}v_{ki}$ are transferred in from markets k, $k \neq i$. After these transfers the amount delivered by industry i to industry j is:

$$u_{ij} - \sum_{k\neq i} b_{jik}v_{ik} + \sum_{k\neq j} b_{jki}v_{ki} \qquad (2)$$

Intuiteively, (2) might be also derived by transposing the intermediate supply matrix given in figure 2. By this way, a similar structure will come out interchanging i and j subindexes and by transposing the corresponding subindexes of v in (1).

3. Product-by-product coefficients

A product-by-product SIOT consists of input-output coefficients a_{ij}, which measure the amounts of product i required per unit of product j. Now the total input of product i used to make product j is given by formula (1) and the total output of product j is $\sum_k v_{jk}$. Simple division yields the product-by-product input-output coefficient:

$$a_{ij} = (u_{ij} - \sum_{k\neq j} a_{ijk}v_{kj} + \sum_{k\neq j} a_{ikj}v_{jk})/\sum_k v_{jk} \qquad (3)$$

As shown in ten Raa and Rueda-Cantuche (2007), the *product technology assumption* postulates that all products have unique input structures, irrespective the industry of fabrication:

$$a_{ijk} = a_{ik} \tag{4}$$

and reduces equation (3) after some tedious algebra (see Appendix) to:

$$\mathbf{A} = \mathbf{U}\mathbf{V}^{-1} \tag{5}$$

The supply table needs to be square to compute its inverse and negatives may emerge from this operation.

The *industry technology assumption* postulates that all industries have unique input structures irrespective the commodity composition of their produce:

$$a_{ijk} = a_{ij} \tag{6}$$

and reduces equation (3) to (see Appendix):

$$\mathbf{A} = \mathbf{U}\left(\widehat{\mathbf{V}^{\mathsf{T}}\mathbf{e}}\right)^{-1}\mathbf{V}^{\mathsf{T}}\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1} \tag{7}$$

where $^{\mathsf{T}}$ denotes transposition, **e** the summation vector with all entries one, and ^ a diagonal matrix. In this case the supply table does not need to be square and negatives do not emerge.

4. Industry-by-industry coefficients

An industry-by-industry SIOT consists of input-output coefficients b_{ij}, which measure the supplies of industry *i* required per unit of output of industry *j*. Now the total delivery of industry *i* to industry *j* is given by formula (2) and the total output of industry j is $\sum_k v_{kj}$. Simple division yields the industry-by-industry input-output coefficient:

$$b_{ij} = (u_{ij} - \sum_{k\neq i} b_{jik}v_{ik} + \sum_{k\neq j} b_{jki}v_{ki})/\sum_k v_{kj} \qquad (8)$$

Here the *industry technology assumption* postulates that all industries have a unique input structure, irrespective the product market:

$$b_{jik} = b_{jk} \qquad (9)$$

and reduces equation (8) after some tedious algebra (see Appendix) to:

$$\mathbf{A} = \left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)\mathbf{V}^{\mathbf{-1}}\mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} \qquad (10)$$

The supply table needs to be square to compute its inverse and negatives may emerge from this operation.

And here the *product technology* assumption postulates that all products require unique industry deliveries, irrespective the industry of fabrication:

$$b_{jik} = b_{ji} \qquad (11)$$

and reduces equation (8) after some tedious algebra (see Appendix) to:

$$\mathbf{A} = \mathbf{V}^{\mathbf{T}}\left(\widehat{\mathbf{Ve}}\right)^{\mathbf{-1}}\mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} \qquad (12)$$

5. Tests for the product and industry technology models

Following Mattey and ten Raa (1997), we consider input-output coefficients regression coefficients of inputs on outputs given firm data. Thus, let $l = 1, \ldots, m$ ($> n$)[1] be the total number of firms considered while being m_1 the number of firms populating industry 1, m_2, those populating industry 2, … so that $m = m_1 + m_2 + \ldots m_n$. Then, regress each input i on industry j's outputs:

$$u_{ijl} = \sum_{k=1}^{n} a_{ijk} v_{kjl} + \varepsilon_{ijl} = a_{ij1} v_{1jl} + \ldots + a_{ij\mathrm{n}} v_{\mathrm{n}jl} + \varepsilon_{ijl} \tag{13}$$

where u_{ijl} and v_{kjl} are the input i and the outputs k of industry j's firm l. Now, under the product technology hypothesis (4), equation (13) becomes:

$$u_{ijl} = \sum_{k=1}^{n} a_{ik} v_{kjl} + \varepsilon_{ijl} = a_{i1} v_{1jl} \ldots + a_{i\mathrm{n}} v_{\mathrm{n}jl} + \varepsilon_{ijl} \tag{14}$$

Next, by testing the following null hypothesis:

$$H_0 : a_{ik} = a_{ik}^{P},\ \ k = 1, 2, \ldots, n$$

being a_{ik}^{P} the i-th row of equation (5), an empirical test can be carried out using the standard F-statistic to test the statistical significance of the product technology assumption. Standard econometric analysis is in order by means of the so-called p-value, which can be defined as the minimum significance level to reject the null hypothesis (e.g. the product technology assumption). For example, if a p-value equals 0.2, then the imposition of the product technology assumption pushes the error terms of

[1] Notice that m should be greater than n (number of products) to get positive degrees of freedom in the econometric model. Besides, m might be also referred to a more detailed breakdown of n industries and not necessarily to firm micro data, which would be the highest detail level.

equation (13) in the tail with 20% mass, i.e. we shall accept the product technology assumption.

Similarly, under the industry technology hypothesis (6), equation (13) becomes:

$$u_{ijl} = \sum_{k=1}^{n} a_{ij} v_{kjl} + \varepsilon_{ijl} = a_{ij} v_{1jl} + \ldots + a_{ij} v_{njl} + \varepsilon_{ijl} = a_{ij} \sum_{k=1}^{n} v_{kjl} + \varepsilon_{ijl} \tag{15}$$

However, since the resulting equation is only valid for industry *j*, a pooled regression is then proposed as follows:

$$u_{ijl} = a_{i1} \sum_{k=1}^{n} v_{k1l} + \ldots + a_{in} \sum_{k=1}^{n} v_{knl} + \varepsilon_{ijl} \tag{16}$$

where the number of observations would be m as well. The first m_1 observations of the first term on the right hand side of equation (16) would stand for total outputs of firms from industry 1 while the rest (up to m) should be filled with zeros. In case of the next explanatory variable, the first m_1 observations shall be filled with zeros while the second m_2 observations will provide total outputs of firms that belong to industry 2. Similarly, the remaining data (up to m) should be filled again with zeros. This pooled data construction method can be easily extended further on to n industries, thus building the complete matrix of explanatory variables. Lastly, the dependent variable would represent intermediate uses of input i by industries $j = 1, 2, \ldots, n$.

In this sense, by testing the following null hypothesis:

$$H_0 : a_{ij} = a_{ij}^{I}, \;\; j = 1, 2, \ldots, n$$

where a_{ij}^{I} is the i-th row of equation (7), the F-statistic can be used this time to test the statistical significance of the industry technology assumption. For example, if we have now a p-value equal to 0.3, the imposition of the industry technology assumption pushes the error terms of (13) less, in the tail with 30% mass. In general, a greater p-value indicates a better fit of the technology assumption to the data.

Since the input i has been fixed in this regression analysis, for some inputs the product technology assumption may prove better and for other inputs the industry technology model.

With industry-by-industry tables we proceed the same way. Let l = 1, …, p (> n)[2] be the number of products. Regress industry j's firm consumption of products i = 1, 2, …, p on commodity i's total outputs by industry:

$$u_{ijl} = \sum_{k=1}^{n} b_{jik} v_{ikl} + \varepsilon_{ijl} = b_{ji1} v_{i1l} + \ldots + b_{jin} v_{inl} + \varepsilon_{jil} \tag{17}$$

where u_{ijl} represents industry j's firm intermediate uses of inputs i = 1, 2, …, p and v_{ikl}, product outputs of industries k = 1, 2,…, n. Now, it is under the industry technology hypothesis (9), equation (17) becomes:

$$u_{ijl} = \sum_{k=1}^{n} b_{jk} v_{ikl} + \varepsilon_{ijl} = b_{j1} v_{i1l} + \ldots + b_{jn} v_{inl} + \varepsilon_{jil} \tag{18}$$

and by testing the following null hypothesis:

$$H_0 : b_{jk} = b_{jk}^{I},\ \ k = 1,\ldots,n$$

the industry technology assumption can be tested. Here b_{jk}^{I} is the j-th row of equation (10).

[2] Notice that p shall be greater than n (number of industries) to get positive degrees of freedom. Moreover, p regards with a more detailed breakdown of commodities, remaining the number of industries unchanged.

Finally, equation (12) shows the joint null hypothesis involving the product technology. Under the restriction of equation (11), equation (17) becomes:

$$u_{ijl} = \sum_{k=1}^{n} b_{ji} v_{ikl} + \varepsilon_{ijl} = b_{ji} v_{i1l} + \ldots + b_{ji} v_{\text{in}l} + \varepsilon_{ijl} = b_{ji} \sum_{k=1}^{n} v_{ikl} + \varepsilon_{ijl} \tag{19}$$

Analogously to equation (15), equation (19) is now only valid for input *i*, so a new pooled regression can be postulated as:

$$u_{ijl} = b_{j1} \sum_{k=1}^{n} v_{1kl} + \ldots + b_{j\text{n}} \sum_{k=1}^{n} v_{\text{n}kl} + \varepsilon_{ijl} \tag{20}$$

where the number of observations would be *p*. The first p_1 observations of the first term on the right of equation (20) would be the total firms' output of product 1 (irrespective of the industry where it is produced) being the rest of elements (up to *p*) zeros. For the next term, the first p_1 observations shall be filled with zeros while the second p_2 observations would stand for the total firms' output of product 2. Similarly, the remaining data (up to *p*) shall be a column vector of zeros, again. This pooled data for regression analysis can also be easily extended to *p* products, hence building a full matrix of independent variables. In addition, the dependent variable depicts intermediate uses of inputs $i = 1, 2, \ldots, p$ by industry *j*.

In this sense, by testing the following null hypothesis:

$$H_0 : b_{ji} = b_{ji}^{P},\; i = 1, \ldots, n$$

being b_{ji}^{P} the *j*-th row of equation (12), the *F*-statistic can also be used to test the statistical significance of the product technology assumption.

Eventually, since industry *j* was fixed this time, for some industries the fixed product sales structure assumption may fit better data and for others the fixed industry sales structure hypothesis.

6. Conclusion

The conflict between products and industries in input-output analysis manifests itself at two, independent levels, namely the dimension of the table one wants to construct and the method of construction. For both types of tables we have presented transfer formulas that encompass the alternative methods of construction. Standard econometric tests can be used to determine which method best fits the data. We plan to apply the methodology to the Andalusian accounts. The transfer framework also suggests an axiomatic foundation of the industry technology model which we also plan to provide. At least in principle this paper justifies the dissident statistical practices in Canada, Denmark, Finland, the Netherlands, and Norway.

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Appendix

Product by product tables (product technology)

Under equation (4), equation (3) reads:

$$a_{ij} = (u_{ij} - \textstyle\sum_{k \neq j} a_{ik}v_{kj} + \sum_{k \neq j} a_{ij}v_{jk})/\sum_k v_{jk} \qquad \text{(A.1)}$$

which, in matrix terms, implies that

$$\mathbf{U} = \mathbf{AV}$$

and, consequently:

$$\begin{aligned} \mathbf{A} &= \left(\mathbf{U} - \mathbf{A}\widetilde{\mathbf{V}} + \mathbf{A}\left(\widehat{\widetilde{\mathbf{Ve}}}\right)\right)\left(\widehat{\mathbf{Ve}}\right)^{-1} = \\ &= \left(\mathbf{U} - \mathbf{A}\left(\mathbf{V} - \widehat{\mathbf{V}}\right) + \mathbf{A}\left(\widehat{\mathbf{Ve}}\right) - \mathbf{A}\widehat{\mathbf{V}}\right)\left(\widehat{\mathbf{Ve}}\right)^{-1} = \\ &= \left(\mathbf{U} - \mathbf{AV} + \mathbf{A}\left(\widehat{\mathbf{Ve}}\right)\right)\left(\widehat{\mathbf{Ve}}\right)^{-1} = \mathbf{U}\left(\widehat{\mathbf{Ve}}\right)^{-1} - \mathbf{AV}\left(\widehat{\mathbf{Ve}}\right)^{-1} + \mathbf{A} \end{aligned} \qquad \text{(A.2)}$$

Next, by substituting $\mathbf{A = UV^{-1}}$, equation (A.2) yields:

$$\mathbf{U}\left(\widehat{\mathbf{Ve}}\right)^{-1} - \mathbf{AV}\left(\widehat{\mathbf{Ve}}\right)^{-1} + \mathbf{A} = \mathbf{U}\left(\widehat{\mathbf{Ve}}\right)^{-1} - \mathbf{UV^{-1}V}\left(\widehat{\mathbf{Ve}}\right)^{-1} + \mathbf{UV^{-1}} = \mathbf{UV^{-1}}$$

having as final result that:

$$\mathbf{A} = \mathbf{UV}^{-1}$$

Product by product tables (industry technology)

Under equation (6), equation (3) turns out to be:

$$a_{ij} = (u_{ij} - \sum_{k \neq j} a_{ij} v_{kj} + \sum_{k \neq j} a_{ik} v_{jk}) / \sum_k v_{jk} \tag{A.3}$$

which, in matrix form, involves that u_{ij} can be said to be proportional to total industries' outputs:

$$\mathbf{U} = \mathbf{A}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)$$

Analogously to equation (A.2), this time the **A** matrix would be as follows:

$$\mathbf{A} = \left(\mathbf{U} - \mathbf{A}\left(\widehat{\tilde{\mathbf{V}}^{\mathbf{T}}\mathbf{e}}\right) + \mathbf{A}\tilde{\mathbf{V}}^{\mathbf{T}}\right)\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1} \tag{A.4}$$

or, manipulating equation (A.4),

$$\begin{aligned}
\mathbf{A} &= \left(\mathbf{U} - \mathbf{A}\left(\widehat{\tilde{\mathbf{V}}^{\mathbf{T}}\mathbf{e}}\right) + \mathbf{A}\tilde{\mathbf{V}}^{\mathbf{T}}\right)\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1} = \\
&= \left(\mathbf{U} - \mathbf{A}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right) + \mathbf{A}\widehat{\mathbf{V}^{\mathbf{T}}} + \mathbf{A}\left(\mathbf{V}^{\mathbf{T}} - \widehat{\mathbf{V}^{\mathbf{T}}}\right)\right)\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1} = \\
&= \left(\mathbf{U} - \mathbf{A}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right) + \mathbf{A}\mathbf{V}^{\mathbf{T}}\right)\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1} = \mathbf{U}\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1} - \mathbf{A}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1} + \mathbf{A}\mathbf{V}^{\mathbf{T}}\left(\widehat{\mathbf{V}\mathbf{e}}\right)^{-1}
\end{aligned} \tag{A.5}$$

and finally, by substituting

$$\mathbf{A} = \mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{\mathbf{-1}}$$

it is obtained that:

$$\mathbf{A} = \mathbf{U}\left(\widehat{\mathbf{Ve}}\right)^{-1} - \mathbf{A}\left(\widehat{\mathbf{V^T e}}\right)\left(\widehat{\mathbf{Ve}}\right)^{-1} + \mathbf{A}\mathbf{V^T}\left(\widehat{\mathbf{Ve}}\right)^{-1} =$$

$$= \mathbf{U}\left(\widehat{\mathbf{Ve}}\right)^{-1} - \mathbf{U}\left(\widehat{\mathbf{V^T e}}\right)^{-1}\left(\widehat{\mathbf{V^T e}}\right)\left(\widehat{\mathbf{Ve}}\right)^{-1} + \mathbf{U}\left(\widehat{\mathbf{V^T e}}\right)^{-1}\mathbf{V^T}\left(\widehat{\mathbf{Ve}}\right)^{-1} =$$

$$= \mathbf{U}\left(\widehat{\mathbf{V^T e}}\right)^{-1}\mathbf{V^T}\left(\widehat{\mathbf{Ve}}\right)^{-1}.$$

Industry by industry tables (industry technology)

This time, equation (8) becomes into:

$$b_{ij} = (u_{ij} - \textstyle\sum_{k\neq i} b_{jk}v_{ik} + \sum_{k\neq j} b_{ji}v_{ki})/\sum_k v_{kj} \qquad \text{(A.6)}$$

which implicitly assumes that $\mathbf{U} = \mathbf{VB^T}$, and hence:

$$\mathbf{A} = \left(\mathbf{U} - \widetilde{\mathbf{V}}\mathbf{B^T} + \left(\widehat{\widetilde{\mathbf{V^T}}\mathbf{e}}\right)\mathbf{B^T}\right)\left(\widehat{\mathbf{V^T e}}\right)^{-1} =$$

$$= \left(\mathbf{U} - \left(\mathbf{V} - \widehat{\mathbf{V}}\right)\mathbf{B^T} + \left(\widehat{\mathbf{V^T e}}\right)\mathbf{B^T} - \widehat{\mathbf{V}}^{\mathbf{T}}\mathbf{B^T}\right)\left(\widehat{\mathbf{V^T e}}\right)^{-1} = \qquad \text{(A.7)}$$

$$= \left(\mathbf{U} - \mathbf{VB^T} + \left(\widehat{\mathbf{V^T e}}\right)\mathbf{B^T}\right)\left(\widehat{\mathbf{V^T e}}\right)^{-1} = \mathbf{U}\left(\widehat{\mathbf{V^T e}}\right)^{-1} - \mathbf{VB^T}\left(\widehat{\mathbf{V^T e}}\right)^{-1} + \left(\widehat{\mathbf{V^T e}}\right)\mathbf{B^T}\left(\widehat{\mathbf{V^T e}}\right)^{-1}$$

Next, by substituting $\mathbf{B^T = V^{-1}U}$, equation (A.7) would result in:

$$\mathbf{U}\left(\widehat{\mathbf{V^T e}}\right)^{-1} - \mathbf{VB^T}\left(\widehat{\mathbf{V^T e}}\right)^{-1} + \left(\widehat{\mathbf{V^T e}}\right)\mathbf{B^T}\left(\widehat{\mathbf{V^T e}}\right)^{-1} =$$

$$= \mathbf{U}\left(\widehat{\mathbf{V^T e}}\right)^{-1} - \mathbf{VV^{-1}U}\left(\widehat{\mathbf{V^T e}}\right)^{-1} + \left(\widehat{\mathbf{V^T e}}\right)\mathbf{V^{-1}U}\left(\widehat{\mathbf{V^T e}}\right)^{-1} =$$

$$= \left(\widehat{\mathbf{V^T e}}\right)\mathbf{V^{-1}U}\left(\widehat{\mathbf{V^T e}}\right)^{-1} = \mathbf{A}$$

Industry by industry tables (product technology)

Finally, equation (8) turns out to be:

$b_{ij} = (u_{ij} - \sum_{k \neq i} b_{ji} v_{ik} + \sum_{k \neq j} b_{jk} v_{ki}) / \sum_k v_{kj}$ (A.8)

which involves:

$$\mathbf{U} = \left(\widehat{\mathbf{Ve}}\right)\mathbf{B}^{\mathbf{T}}$$

thus being u_{ij} proportional to total commodity outputs. Therefore,

$$\begin{aligned} \mathbf{A} &= \left(\mathbf{U} - \left(\widehat{\widetilde{\mathbf{Ve}}}\right)\mathbf{B}^{\mathbf{T}} + \widetilde{\mathbf{V}^{\mathbf{T}}}\mathbf{B}^{\mathbf{T}} \right)\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} = \\ &= \left(\mathbf{U} - \left(\widehat{\mathbf{Ve}}\right)\mathbf{B}^{\mathbf{T}} + \hat{\mathbf{V}}\mathbf{B}^{\mathbf{T}} + \left(\mathbf{V}^{\mathbf{T}} - \widehat{\mathbf{V}^{\mathbf{T}}}\right)\mathbf{B}^{\mathbf{T}}\right)\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} = \qquad \text{(A.9)} \\ &= \left(\mathbf{U} - \left(\widehat{\mathbf{Ve}}\right)\mathbf{B}^{\mathbf{T}} + \mathbf{V}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\right)\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} = \mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} - \left(\widehat{\mathbf{Ve}}\right)\mathbf{B}^{\mathbf{T}}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} + \mathbf{V}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} \end{aligned}$$

and once again, by substituting:

$$\mathbf{B}^{\mathbf{T}} = \left(\widehat{\mathbf{Ve}}\right)^{\mathbf{-1}}\mathbf{U}$$

then, equation (A.9) would yield:

$$\begin{aligned} &\mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} - \left(\widehat{\mathbf{Ve}}\right)\mathbf{B}^{\mathbf{T}}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} + \mathbf{V}^{\mathbf{T}}\mathbf{B}^{\mathbf{T}}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} = \\ &= \mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} - \left(\widehat{\mathbf{Ve}}\right)\left(\widehat{\mathbf{Ve}}\right)^{\mathbf{-1}}\mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} + \mathbf{V}^{\mathbf{T}}\left(\widehat{\mathbf{Ve}}\right)^{\mathbf{-1}}\mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} = \\ &= \mathbf{V}^{\mathbf{T}}\left(\widehat{\mathbf{Ve}}\right)^{\mathbf{-1}}\mathbf{U}\left(\widehat{\mathbf{V}^{\mathbf{T}}\mathbf{e}}\right)^{-1} = \mathbf{A} \end{aligned}$$

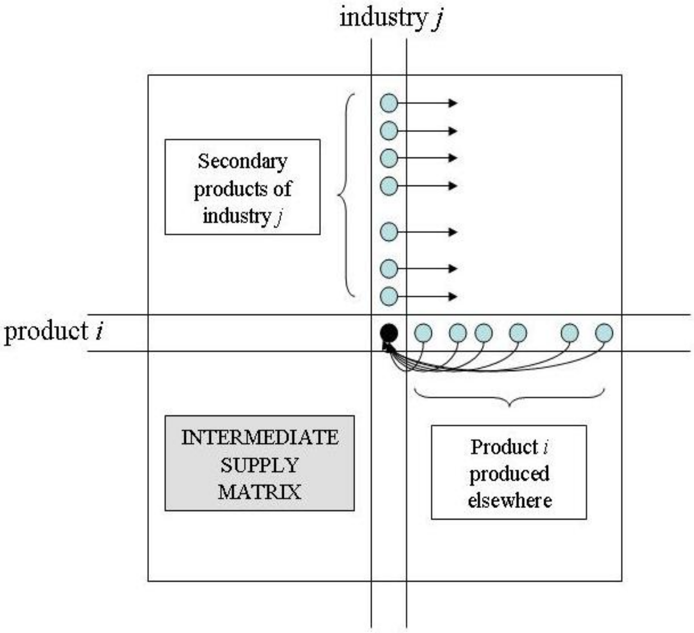

Figure 1: Transfers for a product-by-product table

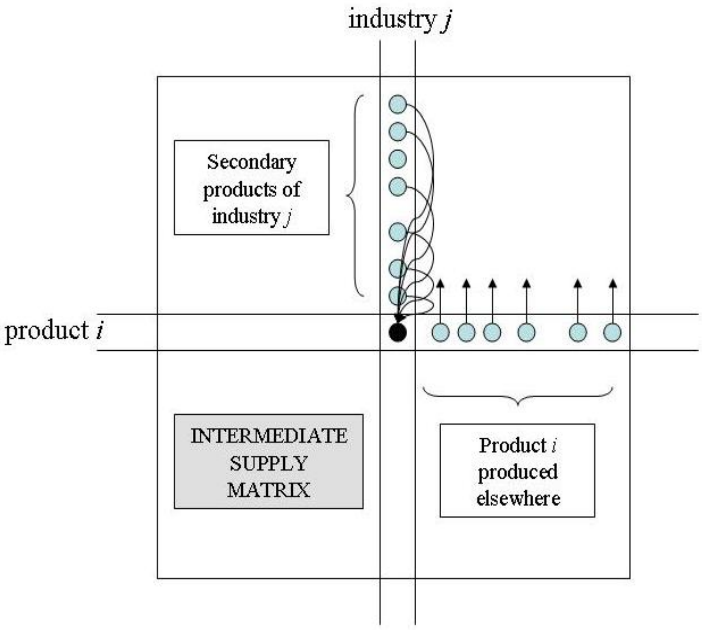

Figure 2: Transfers for an industry-by-industry table